Tutorial 12: Selected problems of Assignment 11

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27/11/2019

Baire Category Theorem

Thm Let (X,d) be a complete metric space, then for any countable set of nowhere dense sets $\{E_n \mid n \in \mathbb{N}\}$, $\bigcup_{n\in\mathbb{N}} E_n$ has empty interior.

In words: any set in X of the first category has empty interior.

Equivalently: Any residual set is dense.

Cor Under same assumptions, if $X = \bigcup_{n\in\mathbb{N}} F_n$ for F_n: closed, then $\exists\, n \in \mathbb{N}$ s.t. F_n has nonempty interior.

Q1) (HW11, Q6) Let (X,d) be a complete metric space, and $\mathcal{F} \subseteq C(X)$ be pointwisely bdd: $\forall x \in X, \exists M \in \mathbb{R}$ s.t. $|f(x)| \leq M, \forall f \in \mathcal{F}$.

Show that $\mathcal{F}$ is "somewhere uniformly bdd":

$\exists \phi \neq G \subseteq X$ open, $\exists C \in \mathbb{R}$ s.t. $\forall x \in G, \forall f \in \mathcal{F}, |f(x)| \leq C$.

Sol) $\forall n \in \mathbb{N}$, define $F_n = \{x \in X \mid |f(x)| \leq n, \forall f \in \mathcal{F}\}$.

Then F_n is closed by continuity of f, and $X = \bigcup_{n \in \mathbb{N}} F_n$ by ptwise bdd property of $\mathcal{F}$.

Therefore, by Cor, $\exists n_1 \in \mathbb{N}$ s.t. $\mathrm{Int}(X_{n_1}) \neq \phi$

Define $G = \mathrm{Int}(X_{n_1})$, $C = n_1$, then by definition,

$\forall x \in G, \forall f \in \mathcal{F}, |f(x)| \leq C$. $\blacksquare$

Q2) (HW11, Q7) Fix $I=[0,1]\subseteq\mathbb{R}$: $f\in C(I)$ is called non-monotonic if

$\forall J$: closed subinterval of I of positive length, f is not monotonic on J.

Show that $\mathcal{N}:=\{f\in C(I)\mid f \text{ is non-monotonic}\}$ is dense in $C[a,b]$.

Sol) By Thm, it suffices to show that $\mathcal{N}$ is residual.

Let $A:=\{(x,n)\in I\times\mathbb{N}\mid x\in\mathbb{Q};\ x\neq 0,1\}$, then A is countable ;

$\forall (x,n)\in A$, define $\mathcal{E}_{x,n}=\{f\in C(I)\mid \forall y\in \overline{B_{\frac{1}{n}}(x)}\cap I,\ (f(y)-f(x))(y-x)\geq 0\}$

and $\mathcal{F}_{x,n}=\{f\in C(I)\mid \forall y\in \overline{B_{\frac{1}{n}}(x)}\cap I,\ (f(y)-f(x))(y-x)\leq 0\}$

Then $f\notin\mathcal{N}\Leftrightarrow f$ is not non-monotonic

$\Leftrightarrow \exists J\subseteq I$ as above f is monotonic over J

$\Leftrightarrow \exists (x,n)\in A$ s.t. $f\in\mathcal{E}_{x,n}\cup\mathcal{F}_{y,n}$

$$\therefore\ C(I)\setminus\mathcal{N}=\bigcup_{(x,n)\in A}(\mathcal{E}_{x,n}\cup\mathcal{F}_{x,n})$$

Hence, it suffices to show that $\forall (x,n)\in A$, $\mathcal{E}_{x,n}\cup\mathcal{F}_{x,n}$ is nowhere dense.

In what follows, we will show that $\mathcal{E}_{x,n}$ and $\mathcal{F}_{x,n}$ are nowhere dense:

(i) $\mathcal{E}_{x,n}$ is nowhere dense : We first show that $\mathcal{E}_{x,n}$ is closed :

$\forall (f_k) \subseteq \mathcal{E}_{x,n}$ converging to f, showing $f \in \mathcal{E}_{x,n}$:

By assumption, $\forall k \in \mathbb{N}$, $\forall y \in \overline{B_{\frac{1}{n}}(x)} \cap I$, $(f_k(y) - f_k(x))(y-x) \geq 0$

$\therefore \quad (f(y)-f(x))(y-x) = \lim_{k\to\infty} (f_k(y) - f_k(x))(y-x) \geq 0$

$\therefore \quad f \in \mathcal{E}_{x,n}$, $\therefore$ $\mathcal{E}_{x,n}$ is closed.

Showing $\mathcal{E}_{x,n}$ is nowhere dense by definition :

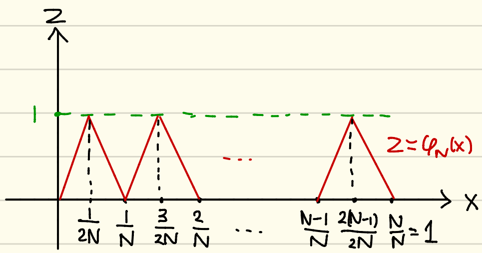
$\forall f \in \overline{\mathcal{E}_{x,n}} = \mathcal{E}_{x,n}$, $\forall \varepsilon > 0$, showing $B_\varepsilon(f) \not\subseteq \mathcal{E}_{x,n}$:

By Weierstrass approximation Thm, $\exists$ polynomial p s.t. $p \in B_{\frac{\varepsilon}{2}}(f)$;

since $p|_I$ is C^1, it is Lipschitz continuous with constant L.

$\forall N \in \mathbb{N}$, define $\varphi_N : I = [0,1] \longrightarrow \mathbb{R}$ jig-saw function

which is piecewise linear, $\frac{1}{N}$-periodic with slopes $\pm 2N$:



Define $g_N(x) = p(x) + \frac{\varepsilon}{2}\cdot\varphi_N(x)$. Then $g_N \in C(I)$

(1) $g_N \in B_\varepsilon(f)$: $\|g_N - f\|_\infty = \|(p-f) + \frac{\varepsilon}{2}\varphi_N\|_\infty < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

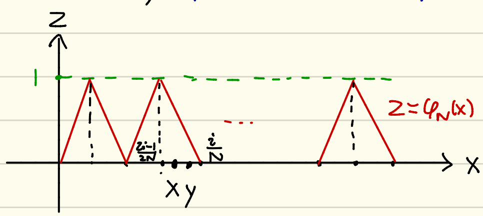
(2) $g_N \notin E_{x,n}$ for some N : $\forall y \in I$ with $y > x$.

$$(g_N(y) - g_N(x))(y-x) = \left((p(y)-p(x)) + \frac{\varepsilon}{2}(\varphi_N(y) - \varphi_N(x))\right)(y-x)$$

$$\leq \left(L(y-x) + \frac{\varepsilon}{2}(\varphi_N(y) - \varphi_N(x))\right)(y-x)$$

Choose $N \in \mathbb{N}$ satisfying $\begin{cases} N > \frac{L}{\varepsilon} \\ \frac{2i-1}{2N} \leq x < \frac{i}{N}, \ \exists\, i \in \mathbb{N};\ 1 \leq i \leq N \end{cases}$

Choose any $y \in I$ with $x < y < \frac{i}{N}$ and $y - x \leq \frac{1}{n}$



then by definition $\varphi_N(y) - \varphi_N(x) = (-2N)(y-x)$

$\therefore (g_N(y) - g_N(x))(y-x) \leq (L(y-x) - N\varepsilon(y-x))(y-x) = (L - N\varepsilon)(y-x)^2 < 0$

$\therefore g_N \notin E_{x,n}$

Therefore, $E_{x,n}$ is nowhere dense.

(ii) $\tilde{F}_{x,n}$ is nowhere dense : Similar argument as in (i)

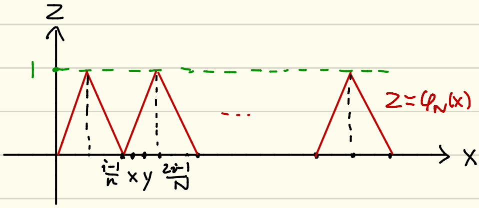
except to show (2):

(2) $g_N \notin \tilde{F}_{x,n}$ for some N : $\forall y \in I$ with $y > x$,

$(g_N(y) - g_N(x))(y-x) \geqslant (L(y-x) + \frac{\varepsilon}{2}(\varphi_N(y) - \varphi_N(x)))(y-x)$

Choose $N \in \mathbb{N}$ satisfying $\begin{cases} N > \frac{L}{\varepsilon} \\ \frac{i-1}{N} \leqslant x < \frac{2i-1}{2N}, \ \exists\, i \in \mathbb{N};\ 1 \leq i \leq N \end{cases}$

Choose any $y \in I$ with $x < y < \frac{2i-1}{2N}$ and $y - x \leq \frac{1}{n}$



then by definition $\varphi_N(y) - \varphi_N(x) = 2N(y-x)$

$\therefore (g_N(y) - g_N(x))(y-x) \geq (-L(y-x) + N\varepsilon(y-x))(y-x) = (N\varepsilon - L)(y-x)^2 > 0$

$\therefore g_N \notin F_{x,n}$

Therefore, $F_{x,n}$ is nowhere dense.